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Gravitational Quadrupole Radiation of Angular Momentum

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We obtain the quadrupole angular momentum radiation of gravity from the recently obtained covariant conservation law of angular momentum. Our result agrees with that derived from the Landau–Lifshitz energy-momentum pseudotensor.

1. INTRODUCTION

Conservation laws of energy-momentum and angular momentum have been of fundamental interest in gravitational physics (Penrose, 1982). Using the vierbein representation of general relativity, Duan and Zhang (1963) obtained a general covariant conservation law of energy-momentum which overcomes the difficulties of other expressions (Duan *et al.*, 1988). This conservation law gives the correct quadrupole radiation formula of energy (Duan and Wang, 1983), which is in good agreement with the analysis of the gravitational damping for the pulsar PSR1916+13. On the other hand, the conservation law of angular momentum has also been discussed in different approaches. Landau and Lifshitz (1987) and Fock (1959) established their angular momentum conservation laws from symmetric pseudotensors, which are therefore noncovariant. In another approach, Komar (1959) suggested some integrals, and following this, Ashtekar and Winicour (1982) introduced some linkages, but these definitions involve some ambiguities (Penrose, 1982). Recently, Duan and Feng (n.d.; Feng and Duan, n.d.) proposed a covariant conservation law of angular momentum which does not suffer from the difficulties of the others. The corresponding conservative angular

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momentum for some specific spacetimes shows that this conservation law is reasonable.

In this paper, we derive the gravitational radiation of angular momentum from the general covariant conservation law (Duan and Feng, n.d.; Feng and Duan, n.d.). As expected, the radiation is related to that of energy and depends on the quadrupole. In Section 2, we briefly review the general covariant conservation law of angular momentum. In Section 3, we derive the radiation from the weak-field approximation and make a few remarks.

2. GENERAL COVARIANT ANGULAR MOMENTUM CONSERVATION LAW

In deriving the general covariant conservation law of energy-momentum in general relativity (Duan and Zhang, 1963) the general displacement transformation, which is a generalization of the displacement transformation in Minkowski spacetime, was used. In the local Lorentz reference frame, the general displacement transformation takes the same form as that in Minkowski spacetime. This implies that general covariant conservation laws correspond to the invariance of the action under local transformations. We may conjecture that since the conservation law for angular momentum in special relativity corresponds to the invariance of the action under the Lorentz transformation, the general covariant conservation law of angular momentum is general relativity may be obtained by means of local Lorentz invariance. This conjecture proves to be reasonable (Duan and Feng, n.d.; Feng and Duan, n.d.).

In general relativity, the total action of the gravity–matter system is expressed as (Landau and Lifshitz, 1987)

$$I = \int_M \mathcal{L}\, d^4x = \int_M (\mathcal{L}_g + \mathcal{L}_m)\, d^4x \tag{1}$$

$$\mathcal{L}_g = \frac{c^4}{16\pi G} \sqrt{-g}\, g^{\alpha\beta}(\Gamma^{\nu}_{\mu\alpha}\Gamma^{\mu}_{\nu\beta} - \Gamma^{\sigma}_{\mu\sigma}\Gamma^{\mu}_{\alpha\beta}) \tag{2}$$

where $\Gamma^{\nu}_{\mu\alpha}$ are the Christoffel symbols, $\mathcal{L}_m$ is the matter part of the Lagrangian, and G is the Newtonian gravitational constant. We use the vierbein description; our notations are as follows: e^a_μ are the vierbein components and e^μ_a their inverses, $g_{\mu\nu} = \eta_{ab}e^a_\mu e^b_\nu$, $\eta_{ab} = (1, -1, -1, -1)$, $\omega_{\mu ab}$ are the spin connections which are defined by

$$\mathcal{D}_\mu e^a_\nu \equiv \partial_\mu e^a_\nu - \omega_{\mu ab}\epsilon^b_\nu - \Gamma^{\lambda}_{\mu\nu}e^a_\lambda = 0 \tag{3}$$

and $\omega_{abc} = e^\mu_a \omega_{\mu bc}$ and $\omega_a = \eta^{bc}\omega_{bac}$. It can be proved that

$$\mathcal{L}_g = \frac{c^4}{16\pi G}\, e(D_\mu e^\nu_a D_\nu e^{a\mu} - e^{a\nu} e^b_\lambda D_\nu e^\lambda_a D_\sigma e^\sigma_b) \tag{4}$$

$$\mathcal{L}_g = \mathcal{L}_w - \frac{c^4}{16\pi G}\,\Delta \tag{5}$$

$$\mathcal{L}_w = \frac{c^4}{16\pi G}\,(\omega_a \omega^a - \omega_{abc}\omega^{cba}) \tag{6}$$

$$\Delta = \partial_\mu(e e^{a\mu}\partial_\nu e^\nu_a - e e^\nu_a \partial_\nu e^{a\mu}) \tag{7}$$

where

$$D_\mu e^\nu_a \equiv \partial_\mu e^\nu_a - \omega_{\mu ab} e^{\nu b}$$
$$e = \sqrt{-g} \tag{8}$$

and Δ is a divergence term.

The local vierbein Lorentz transformation takes the form

$$e^a_\mu(x) \to e'^a_\mu = \Lambda^a{}_b(x) e^b_\mu(x)$$
$$\eta_{ab}\Lambda^a{}_c(x)\Lambda^b{}_d(x) = \eta_{cd} \tag{9}$$

It is required that $\mathcal{L}_m$ is invariant under (9), and $\mathcal{L}_g$ is invariant obviously. So under the transformation (9), $\mathcal{L}$ is invariant, i.e.,

$$[\mathcal{L}]_{e^\nu_a}\delta e^\nu_a + [\mathcal{L}]_{\phi^A}\delta\phi^A + \partial_\mu\left(\frac{\partial\mathcal{L}}{\partial\partial_\mu e^\nu_a}\,\delta e^\nu_a + \frac{\partial\mathcal{L}}{\partial\partial_\mu\phi^A}\,\delta\phi^A\right) = 0 \tag{10}$$

where $[\mathcal{L}]_{e^\nu_a}$ and $[\mathcal{L}]_{\phi^A}$ are the Euler expressions defined as

$$[\mathcal{L}]_{e^\nu_a} = \frac{\partial\mathcal{L}}{\partial e^\nu_a} - \partial_\mu\,\frac{\partial\mathcal{L}}{\partial\partial_\mu e^\nu_a} \tag{11}$$

$$[\mathcal{L}]_{\phi^A} = \frac{\partial\mathcal{L}}{\partial\phi^A} - \partial_\mu\,\frac{\partial\mathcal{L}}{\partial\partial_\mu\phi^A} \tag{12}$$

Using the Einstein equation

$$[\mathcal{L}]_{e^\mu_a} = 0$$

i.e.,

$$[\mathcal{L}_g]_{e^\mu_a} + [\mathcal{L}_m]_{e^\mu_a} = 0$$

and the equation of motion of matter

$$[\mathcal{L}]_{\phi^A} = 0 \tag{13}$$

we get the following by (10):

$$\partial_\mu\left(\frac{\partial \mathcal{L}_g}{\partial \partial_\mu e_a^\nu}\,\delta e_a^\nu\right) + \partial_\mu\left(\frac{\partial \mathcal{L}_m}{\partial \partial_\mu e_a^\nu}\,\delta e_a^\nu + \frac{\partial \mathcal{L}_m}{\partial \partial_\mu \phi^A}\,\delta\phi^A\right) = 0 \tag{14}$$

where we have used the fact that only $\mathcal{L}_m$ contains the matter field ϕ_A ($A = 1, \ldots, N$). Consider the infinitesimal local Lorentz transformation

$$\Lambda^a{}_b(x) = \delta^a{}_b + \alpha^a{}_b(x)$$
$$\alpha_{ab} = -\alpha_{ba} \tag{15}$$

and from (9), we have

$$\delta e_a^\mu(x) = \alpha_{ab}(x) e^{\mu b}(x) \tag{16}$$

Suppose that $\mathcal{L}_m$ takes the form

$$\mathcal{L}_m = \mathcal{L}_m(e_a^\mu, \phi^A, D_\mu\phi^A) \tag{17}$$

The ϕ^A belong to some representation of the Lorentz group with generators I_{ab} ($a, b = 0, 1, 2, 3$), $I_{ab} = -I_{ba}$, and D_μ is the covariant derivative

$$D_\mu\phi^A = \partial_\mu\phi^A - \frac{1}{2}\,\omega_{\mu ab}(I^{ab})^A{}_B\phi^B \tag{18}$$

Then under the transformation (9), ϕ^A transforms as

$$\phi^A(x) \to \phi'^A(x) = [D(\alpha)]^A{}_B\phi^B(x) \tag{19}$$

$D(\alpha)$ can be linearized near the identity when the α_{ab} are infinitesimal,

$$[D(\alpha)]^A{}_B = \delta^A{}_B + \frac{1}{2}\,(I_{ab})^A{}_B\alpha^{ab}(x) \tag{20}$$

Thus under the transformation (9), ϕ^A varies as

$$\delta\phi^A(x) = \frac{1}{2}\,(I_{ab})^A{}_B\phi^B(x)\alpha^{ab}(x) \tag{21}$$

We introduce J^μ_{ab} such that

$$eJ^\mu_{ab}\alpha^{ab} = \frac{3}{c}\left[\frac{\partial \mathcal{L}_\omega}{\partial \partial_\mu e^{a\nu}}\,e_b^\nu\alpha^{ab} + \frac{\partial \mathcal{L}_m}{\partial \partial_\mu e^{a\nu}}\,e_b^\nu\alpha^{ab} + \frac{\partial \mathcal{L}_m}{\partial \partial_\mu \phi^A}\,\frac{1}{2}\,(I_{ab})^A{}_B\alpha^{ab}\phi^B\right] \tag{22}$$

Then (14) can be written as

$$\partial_\mu(eJ^\mu_{ab}\alpha^{ab}) - \frac{3c^3}{16\pi G}\,\partial_\mu\left(\frac{\partial\Delta}{\partial\partial_\mu e^\nu_a}\,\delta e^\nu_a\right) = 0 \tag{23}$$

From (7) one can get

$$\frac{\partial\Delta}{\partial\partial_\lambda e^\mu_l}\,e^{m\mu}\alpha_{lm} = \alpha^{lm}\partial_\mu(eV^{\mu\lambda}_{lm}) \tag{24}$$

where

$$V^{\mu\lambda}_{lm} = e^\mu_l e^\lambda_m - e^\mu_m e^\lambda_l \tag{25}$$

Substituting (24) into (23), we obtain

$$\partial_\mu(eJ^\mu_{ab}\alpha^{ab}) - \frac{3c^3}{16\pi G}\,\partial_\mu[\alpha^{ab}\partial_\nu(eV^{\nu\mu}_{ab})] = 0 \tag{26}$$

i.e.,

$$\partial_\mu(eJ^\mu_{ab})\alpha^{ab} + \left[eJ^\mu_{ab} - \frac{3c^3}{16\pi G}\,\partial_\nu(eV^{\nu\mu}_{ab})\right]\partial_\mu\alpha^{ab} = 0 \tag{27}$$

Since α_{ab} and $\partial_\mu\alpha^{ab}$ are independent of each other, we must have

$$\partial_\mu(eJ^\mu_{ab}) = 0 \tag{28}$$

$$J^\mu_{ab} = \frac{3c^3}{16\pi G}\,\nabla_\lambda V^{\lambda\mu}_{ab} \tag{29}$$

or

$$J^\mu_{ab} = \frac{3c^3}{16\pi G}\,(\omega_a e^\mu_b + \omega_{ab}{}^c e^\mu_c - \omega_b e^\mu_a - \omega_{ba}{}^c e^\mu_c) \tag{30}$$

Since $V^{\nu\mu}_{ab}$ is an antisymmetric tensor with respect to the indices μ and ν, this means that J^μ_{ab} is conserved identically. As usual, we call the $V^{\nu\mu}_{ab}$ superpotentials. Since the current J^μ_{ab} is derived from the local Lorentz invariance of the total Lagrangian, it can be interpreted as the angular-momentum tensor density of the gravity–matter system. From (25) and (29) we see that the current J^μ_{ab} of the gravity–matter system is only determined by the vierbein; this feature is quite similar to the theory of the conservation law of energy-momentum in general relativity (Duan and Zhang, 1963). This is because the information of the state of motion of the whole gravity–matter system is contained in the vierbein through the Einstein equations.

For a globally hyperbolic Riemann manifold M, there exist Cauchy surfaces Σ_t foliating M. We choose a submanifold D of M joining any two Cauchy surfaces Σ_{t_1} and Σ_{t_2} so the boundary ∂D of D consists of three parts

Σ_{t_1}, Σ_{t_2}, and A, which is at spatial infinity. For an isolated system, the space-time should be asymptotically flat at spatial infinity, so the vierbein has the following asymptotic behavior:

$$\lim_{r\to\infty}(\partial_\mu e_{a\nu} - \partial_\nu e_{a\mu}) = 0 \tag{31}$$

We can obtain the conservative angular momentum J_{ab} and its radiation

$$J_{ab} = \int_{\Sigma_t} J^{\mu}_{ab} e d\Sigma_\mu \tag{32}$$

$$\frac{\partial}{\partial t} J_{ab}(\sigma) = -c \int_{\partial\sigma} J^{i}_{ab} e ds_i \tag{33}$$

where $ed\Sigma_\mu$ is the covariant surface element of Cauchy surface Σ_t, and $d\Sigma_\mu = 1/3!\ \epsilon_{\mu\nu\alpha\beta} dx^\nu \Lambda dx^\alpha \Lambda dx^\beta$. It can also be shown that J_{ab} is gauge covariant.

3. QUADRUPOLE RADIATION OF ANGULAR MOMENTUM

As in the derivation of the gravitational radiation of energy, we consider the weak field at large distance from the source bodies. The familiar expansion of the metric is of the form $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$, $|h_{\mu\nu}| << 1$. To first order in $h_{\mu\nu}$, the Einstein equations are

$$\Box\phi_{\mu\nu} = -\frac{16\pi G}{c^4} T_{\mu\nu} \tag{34}$$

with coordinate condition

$$\partial_\mu \phi^{\mu}_{\nu} = 0 \tag{35}$$

where $\phi_{\mu\nu} = h_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}h$, $h = h^{\mu}_{\mu}$. The solution to (34) is of the retarded potential form

$$\phi_{\mu\nu}(\mathbf{r}, t) = -\frac{4G}{c^4} \int \frac{T_{\mu\nu}(\mathbf{r}', t - |\mathbf{r} - \mathbf{r}'|/c)}{|\mathbf{r} - \mathbf{r}'|} d^3\mathbf{r}' \tag{36}$$

For large $r = |\mathbf{r}|$, we have approximately

$$\phi_{\mu\nu}(\mathbf{r}, t) = -\frac{4G}{c^4 r} \int T_{\mu\nu}\left(\mathbf{r}', t - \frac{r}{c}\right) d^3\mathbf{r}' \tag{37}$$

The space components ϕ_{ij} (the indices $i, j, k, \ldots$ run over 1, 2, 3) can be further expressed as

$$\phi_{ij}(\mathbf{r}, t) = -\frac{2G}{c^4 r}\frac{\partial^2}{\partial t^2}\int \rho x_i' x_j'\, d^3\mathbf{r}' \tag{38}$$

For later use, we now derive some useful relations of the derivatives of $\phi_{\mu\nu}$. To first order in $1/r$, i.e., first order in $h_{\mu\nu}$, we have

$$\partial_i\phi_{\mu\nu} = \frac{1}{c}\,\dot{\phi}_{\mu\nu}n_i, \qquad n^i = \frac{x^i}{r} \tag{39}$$

where the dot stands for time derivative. From the coordinate condition (35) we have

$$\partial^0\phi_{0\nu} = -\partial^i\phi_{i\nu} = -\frac{1}{c}\,\dot{\phi}_{i\nu}n^i \tag{40}$$

Equations (39) and (40) together give that

$$\partial^0\phi_{00} = -\frac{1}{c}\,\dot{\phi}_{i0}n^i \tag{41}$$

Thus we have

$$\begin{aligned} \partial^0\phi_{00} &= \frac{1}{c}\,\dot{\phi}_{ij}n^i n^j, \qquad & \partial^i\phi_{00} &= \frac{1}{c}\,\dot{\phi}_{jk}n^j n^k n^i \\ \partial^i\phi_{0j} &= -\frac{1}{c}\,\dot{\phi}_{jk}n^i n^k, \qquad & \partial^0\phi_{0j} &= -\frac{1}{c}\,\dot{\phi}_{jk}n^k \\ & & \partial^i\phi_{jk} &= \frac{1}{c}\,\dot{\phi}_{jk}n^i \end{aligned} \tag{42}$$

All these relations hold to first order.

The expansion of the vierbein is taken to be

$$e_{a\mu} = \eta_{a\mu} + \frac{1}{2}f_{\mu a}, \qquad e_a^{\mu} = \delta_a^{\mu} - \frac{1}{2}f_a{}^{\mu} \tag{43}$$

The relation between $h_{\mu\nu}$ and $f_{\mu a}$ can be easily obtained,

$$h_{\mu\nu} = \frac{1}{2}\,(f_{\mu\nu} + f_{\nu\mu}) \tag{44}$$

The first order of the vierbein gauge condition $\nabla_\mu\omega^\mu{}_{ab} = 0$ ensures that

$$f_{\mu\nu} = f_{\nu\mu} \tag{45}$$

Using the relations

$$\omega_{abc} - \omega_{bac} = e_b^\mu e_a^\nu(\partial_\mu e_{\nu c} - \partial_\nu e_{\mu c}) \tag{46}$$

$$\omega_a = -\eta^{bc}e_b^\mu e_a^\nu(\partial_\mu e_{\nu c} - \partial_\nu e_{\mu c}) \tag{47}$$

we can directly calculate that to second order,

$$\omega_{ab}{}^{\mu} - \omega_{ba}{}^{\mu} = \frac{1}{2}\left\{\partial_b h_a{}^{\mu} - \partial_a h_b{}^{\mu} - \frac{1}{2}\left[h^{\beta}{}_a(\partial_b h_{\beta}{}^{\mu} - \partial_{\beta} h_b{}^{\mu}) + h^{\alpha}{}_b(\partial_\alpha h_a{}^{\mu} - \partial_a h_\alpha{}^{\mu}) + h^{\mu c}(\partial_b h_{ac} - \partial_a h_{bc})\right]\right\} \quad (48)$$

$$\omega_a e_b^{\mu} = \frac{1}{2}\,\delta_b^{\mu}\left\{\partial_a h - \partial^c h_{ac} - \frac{1}{2}\left[h_a^d(\partial_d h - \partial^c h_{dc}) + h^{dc}(\partial_a h_{dc} - \partial_d h_{ac})\right]\right\} - \frac{1}{4}\,h_b^{\mu}(\partial_a h - \partial_c h_a^c) \quad (49)$$

Since the radiation should be proportional to G, it is evident from the expression (30) that only the second-order part of J^i_{ab} contributes to the radiation because it contains G^2, while the first-order part contains G, which will be canceled by the G in the overall constant factor. Practical evaluation can prove this point. Hence we need only the second-order part of J^i_{ab}, which we call K^i_{ab}:

$$K^i_{ab} = -\frac{3G^3}{64\pi G}\left\{\phi_a^c(\partial_b\phi_c^i - \partial_c\phi_b^i) + \frac{1}{2}\,\phi_a^c\delta_b^i\partial_c\phi + \frac{1}{2}\,\phi\partial_a\phi_b^i - \phi_c^i\partial_b\phi_a^c + \delta_b^i\phi_c^d(\partial_a\phi_d^c - \partial_d\phi_a^c) - \frac{1}{2}\,\phi_b^i\partial_a\phi\right\} - (a \leftrightarrow b) \quad (50)$$

Using the relations (42), we obtain

$$\begin{aligned} K^i_{jk}n_i = -\frac{3c^2}{64\pi G}\Big\{ & \phi_j^l(\dot\phi_l^i n_k - \dot\phi_k^i n_l)n_i + \phi_j^0(-\dot\phi_l^i n^l n_k - \dot\phi_k^i)n_i \\ & + \frac{1}{2}\,\phi_j^i n_k(\dot\phi_{pq}n^p n^q n_l + \dot\phi_p^p n_l) + \frac{1}{2}\,\phi\dot\phi_k^i n_i n_j \\ & - \phi_l^i\dot\phi_j^l n_k n_i + \phi_0^i\dot\phi_{jp}n^p n_k n_i \\ & + n_k[\phi_0^0(\dot\phi_{pq}n^p n^q n_j + \dot\phi_{jl}n^l) + \phi_0^l(-\dot\phi_{lp}n^p n_j + \dot\phi_{jp}n^p n_l) \\ & + \phi_l^0(-\dot\phi_p^l n^p n_j - \dot\phi_j^l) + \phi_p^q(\dot\phi_q^p n_j - \dot\phi_j^p n_q)] \\ & - \frac{1}{2}\,\phi_k^i n_i(\dot\phi_{pq}n^p n^q n_j + \dot\phi_p^p n_j)\Big\} - (j \leftrightarrow k) \end{aligned} \quad (51)$$

Making use of the integrals

$$\frac{1}{4\pi}\int n_i n_j \, d\Omega = -\frac{1}{3}\eta_{ij} \tag{52}$$

$$\frac{1}{4\pi}\int n_j n_k n_l n_m \, d\Omega = \frac{1}{15}(\eta_{jk}\eta_{lm} + \eta_{jl}\eta_{km} + \eta_{jm}\eta_{kl}) \tag{53}$$

we have

$$\frac{d}{dt}J_{mn} = \frac{2G}{45c^3}(\dddot{D}^l_m \ddot{D}_{ln} - \dddot{D}^l_n \ddot{D}_{lm}) \tag{54}$$

where D_{ij} is the usual quadrupole moment

$$D_{ij} = \int \rho(3x_i x_j - \eta_{ij}x^p x_p)\, d^3x \tag{55}$$

or

$$\frac{d}{dt}M^l = \frac{1}{2}\epsilon^{lmn}\frac{d}{dt}J_{mn} = \frac{2G}{45c^3}\epsilon^{lmn}\dddot{D}^p_m \ddot{D}_{pn} \tag{56}$$

This is exactly what is given in Landau and Lifshitz (1987).

Finally, we remark that general covariance is a fundamental demand in general relativity, and the covariant conservation law (Duan and Feng, n.d.; Feng and Duan, n.d.), the reasonableness of which has been further shown by this paper, is much more reliable than laws which are not covariant (Landau and Lifshitz, 1987).

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